Tentamen Metrische Ruimten
30 augustus 2007, 09:00 - 12:00 uur, Examenhal

You can answer the exam in Dutch or English.

1. Consider $\mathbb{R}$ with the standard topology and the subset $H = [0,1] \cup \{2\}$ with the induced subspace topology. For each one of the following subsets find its set of limit points, its closure and its boundary. Which of these sets are open, which are closed and which are compact?
   (a) $(1/4, 3/4) \cup \{2\}$.
   (b) $(1/2, 1]$.
   (c) $[0, 1/2)$.
   (d) $[0, 1/2] \cup \{2\}$.
   (e) $\{1/n : n \in \{2, 3, 4, \ldots\}\}$.
   (f) $\mathbb{Q} \cap H$.
   $\mathbb{Q}$ is the set of rational numbers. All the properties (limit points, closure, boundary, openness, closedness, compactness) should be examined with respect to the induced subspace topology in H. Support your answers by arguments.

2. Consider $\mathbb{R}$ with the Zariski topology. Recall that in the Zariski topology a set $U \subseteq \mathbb{R}$ is open if and only if $\mathbb{R} - U$ is finite or $U = \emptyset$. Prove that in the Zariski topology, all subsets of $\mathbb{R}$ are compact.

3. Let Y be a topological space with a topology $\mathcal{T}_Y$ and consider a map $f : A \to Y$ from the set A to the space Y. Define

$$\mathcal{T}_A = \{f^{-1}(U) : U \in \mathcal{T}_Y\}.$$

   Show that $\mathcal{T}_A$ is a topology on A.

4. Given a topological space T and a subset $H \subset T$, prove that
   (a) H is closed in T if and only if $b(H) \subset H$.
   (b) $b(H) = \emptyset$ if and only if H is open and closed in T.
   $b(H)$ denotes the boundary of H.